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LOGICAL INDEFINITES*

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ABSTRACT

The best extant demarcation of logical constants, due to Tarski, classifies logical constants by invariance properties of their denotations. This classification is developed in a framework which presumes that the denotations of all expressions are definite. However, some indefinite expressions, such as Russell's indefinite description operator η, Hilbert's ε, and abstraction operators such as 'the number of', appropriately interpreted, are logical. I generalize the Tarskian framework in such a way as to allow a reasonable account of the denotations of indefinite expressions. This account gives rise to a principled classification of the denotations of logical and non-logical indefinite expressions. After developing this classification and its application to particular cases in some detail, I show how this generalized framework allows a novel view of the logical status of certain abstraction operators such as 'the number of'. I then show how we can define surrogate abstraction operators directly in higher-order languages augmented with an ε-operator.

1. Introduction

We have known since Tarski's 1936 "The Concept of Logical Consequence" how to develop a precise account of logical truth and consequence given a fixed set of logical expressions. We typically fix these logical constants by enumeration. 'And', 'not', and 'every' are logical constants. 'Square', 'the father of', 'Jack', and 'is larger than' are not. This sort of procedure is more than adequate for most mathematical applications since the consequence relation developed *à la* Tarski on the back of conjunction, negation, identity, and a quantifier or two suffices to characterize a wide array of mathematical structures. When we turn a more philosophical eye towards accurately characterizing the concepts of logical truth and logical consequence as they appear in the informal background logic with which we actually do

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mathematics, things get more complicated. Though our list seems adequate in the main, there is no obvious principled connection between those expressions on the list and those not. Philosophers and logicians have thus attempted to give a principled account of the members of the list that explains why ‘and’ and ‘every’ are logical, why ‘square’ and ‘Bob’ are not, and that settles in a reasonable way disputed cases like ‘is identical to’ and ‘most’. One of these attempts, also initiated by Tarski, has risen almost to the level of widespread acceptance in more mathematical contexts: logical expressions are those whose meaning does not depend on the characteristics of particular objects.

Since logical constants do not depend for their meaning on the characteristics of particular objects, their meaning should not change if we switch objects around or substitute some objects for others. Working within a widely held model of the denotations of various expressions, we can develop a formal analogue of this intuitive constraint, selecting out a set of objects invariant under certain transformations as the potential denotations of logical constants. This framework, though useful, carries with it certain limiting presumptions about the potential meaning of expressions in the language it is modeling. In particular, the framework presumes that the meanings of all expressions are definite in a precise sense spelled out below. This presumption systematically perverts the intended meanings of indefinite expressions such as the English indefinite article. So perverted, indefinite expressions have no hope of being logical constants. However, indefinite expressions are common enough in informal mathematical reasoning and plausible enough as candidate logical constants that our best principled account of logical constants should not exclude them by being built on an inadequately accommodating framework.

My aim in this paper is to amend this framework to allow indefinite expressions a reasonable chance at logicality. The structure of the paper is as follows. In 2, I describe Tarski’s account of logical truth and logical consequence. In 2.1, I turn to describing the standard framework in which the invariance criterion of logicality has its home, show how the invariance criterion classifies the denotations of various expressions as logical and non-logical, and explain the intuition behind the criterion. Section 2.2 is a discussion of the adequacy of this criterion as a classification of the logical status of indefinite expressions. Having seen that the criterion is inadequate by virtue of the presumptions of the framework, I propose in sections 3 and 3.1 a more general framework without such presumptions and show how to extend the invariance criterion to this new setting. Section 4 examines the consequences of this extended criterion for a number of cases: a version of Hilbert’s ε operator [4.1], abstraction operators [4.2], and a new type of abstraction operator defined from ε [4.3]. I close in 5 by summarizing the reasons for accepting my proposed amendment of Tarski’s framework and the resulting criterion of logicality.

2. Tarski on Logical Truth and Logical Consequence

A *logical truth* is a sentence true in virtue of its logical form. A sentence A is a *logical consequence* of some sentences $B_0, \ldots, B_n$ if the truth of A is guaranteed by the truth of $B_0, \ldots, B_n, \ldots$ in virtue of the logical form of A and the logical form of the Bs. (Tarski, A. 1936) gives an analysis of 'true in virtue of logical form' according to which a sentence is a logical truth if and only if every way of reinterpreting the non-logical expressions occurring within A results in a true sentence. Likewise A is a logical consequence of $B_0, \ldots, B_n, \ldots$ if every way of reinterpreting the non-logical expressions occurring within A and the Bs that makes all of the Bs true makes A true. With one small amendment, Tarski's analysis has become the standard account of logical truth and logical consequence.

The amendment concerns the connection between true sentences containing only logical vocabulary and logical truths. We can express claims about how many things there are using only standard logical vocabulary. This means that according to the above account such sentences are logical truths. This has the frustrating upshot that sentences expressing facts which are presumably not capable of being sussed out *a priori* are nonetheless logical truths.[1] Contemporary accounts of logical truth avoid this consequence and others by modifying the above definition like so: a sentence is a logical truth if and only if no matter what things there are, every way of reinterpreting the non-logical expressions occurring within A results in a true sentence. Similarly for logical consequence. The technical details of Tarski's approach and our modern variants are not important for this paper, so I set them aside. Details that matter will be filled in below. Almost all of what I say below can be easily adapted to the older approach and, in fact, the results in 4.3 are even better on that approach.[2]

Tarski's definition depends on a distinction between logical and non-logical expressions. In his 1936 paper, Tarski expresses doubt about whether a precise criterion of logicality for expressions could be found. He suggests that though it might be possible to find "objective" arguments that justified the traditional choices of logical constants — the monadic quantifiers 'every'

[1] See (Hanson, W.H. 1997, section 1) for a discussion of problems with the older style of approach relating to the aprioricity of logic. These concerns, unfortunately, are beyond the scope of this paper. Doing them and other concerns about the nature of logic justice would require much fuller treatment than I can manage here. I hope to address this issue elsewhere.

[2] I also bracket the interesting historical question of whether Tarski had the amended version already in mind, but failed to mention this due to the informality of the paper. See Gomez-Torrente, M. (2000); Hanson, W.H. (1997); Etchemendy, J. (1990) and Sher, G. (1991) for discussions of this matter and Mancosu, P. (2010) for an updated survey of the current evidence on offer.

and 'there is', negation, conjunction, etc. — it might also turn out that his analysis yields only a definition of logical truth relative to a choice of logical constants. In later work Tarski suggests a criterion that distinguishes logical from non-logical *notions* where notions are, in a sense to be spelled out precisely below, the denotations of expressions.[3] His approach is entirely extensional; he does not distinguish between expressions with different meanings that denote the same notion. What he offers can be viewed as a necessary condition for being a logical constant — a logical constant denotes a logical notion — and a necessary and sufficient condition for being a logical notion. This goes some way towards settling the choice of logical constants though, as he notes, it does not fully settle the question about logical truth and logical consequence.

2.1. *Tarski's Invariance Criterion*

To spell out the details of Tarski's criterion, we need to introduce a bit of terminology. We define a *type-symbol* as follows:

- 'e' and 't' are type-symbols.
- If $S_1, \dots, S_n$, and S are type-symbols, $(S_1, S_2, \dots, S_n \Rightarrow S)$ is a type symbol.

Given a set of objects (*a domain*) D, we interpret the type-symbols defined above against D thus:

- e denotes D
- t denotes $\{T, F\}$
- $(S_1, S_2, \dots, S_n \Rightarrow S)$ denotes the set of functions from the Cartesian product of $S_1, \dots, S_n$ to S.

For example, $(e \Rightarrow t)$ denotes the set of functions from D to $\{T, F\}$ and $((e \Rightarrow t) \Rightarrow t)$ the set of functions from functions from D to $\{T, F\}$ to $\{T, F\}$. When the right-hand side of a type-symbol is 't', the members of the denoted type will be characteristic functions. We can be slightly perverse and identify a set with its characteristic function and think of, for example, $(e \Rightarrow t)$ being a set of subsets of D (the power set). Likewise, we can think of $((e \Rightarrow t) \Rightarrow t)$ as the power set of the power set of the domain. Taking the union of all the interpretations of all the type-symbols, we obtain a collection of sets we can call the *type-hierarchy over D*.

[3] See Tarski, A. (1986). Tarski's discussion therein is limited to logical notions. He articulates a clear connection between logical constants and logical notions in Tarski, Alfred and Givant, Steven R. (1987), 57 where, in the context of developing set theory in a variable-free formalism, he gives the denotation of a logical notion as a necessary and sufficient condition for being a logical constant. Thanks to a helpful reviewer.

A *notion-in-extension* σ is a function that assigns, to some domains D, some element σ^D of the type-hierarchy over D.[4] A *total notion* is one defined on every domain. We can understand expressions as picking out notions-in-extension. So, for example, the English quantifiers or the formal symbols $\exists$, $\forall$ pick out notions that assign to every domain D a function of type $((e \Rightarrow t) \Rightarrow t)$. Again being slightly perverse, '$\forall$' picks out the singleton of the domain $\{D\}$ and '$\exists$' the set consisting of non-empty subsets of the domain. Type $\langle 1 \rangle$ generalized quantifiers such as 'There are at least four' and 'There are finitely many' can likewise be treated as picking out subsets of the power set of the domain. Type $\langle 1, 1 \rangle$ quantifiers such as 'as many A as B' are of type $(((e \Rightarrow t), (e \Rightarrow t)) \Rightarrow t)$ and can thus be thought of as subsets of the Cartesian product of the power set with itself.[5] And so on. Notions are presumed to be total unless otherwise noted. Partial notions have a tenuous claim to logicality, failing to have universal application.[6]

Many distinct expressions of natural language will pick out the same notion. 'Universal quantification' picks out the notion that sends a domain to its singleton, but so does 'mock universal quantification' which is universal quantification if there are at least 43 things and existential otherwise.[7] The expressions here differ in meaning, so we are not classifying expressions, whether natural or formal, as logical or non-logical, but only notions.

Tarski's approach to classifying notions as logical proceeds in terms of in-variance of notions under permutations of the domain. As McGee puts it:

> Any operation which is disturbed by a permutation must somehow discriminate among individuals in the domain, and any consideration which discriminates among individuals lies beyond the reach of logic, whose concerns are entirely general. (McGee, V. 1996)

A *permutation* of a set is a bijection (one-to-one correspondence) from it to itself. Given a permutation π of a domain D, we can extend π in a straightforward way to a function π^+ on all members of the type-hierarchy over D. We set $\pi^+(T) = T$, $\pi^+(F) = F$. For all d in D, let $\pi^+(d) = \pi(d)$. For an ordered n-tuple $\langle m_1, \ldots, m_n \rangle$ of members of the type-hierarchy,

$$\pi^+(\langle m_1, \ldots, m_n \rangle) = \langle \pi^+(m_1), \ldots, \pi^+(m_n) \rangle.$$

[4] I will typically abbreviate this to 'notion'.

[5] See Peters, S. and Westerståhl, D. (2006) for a useful overview of generalized quantifiers.

[6] This point will be discussed further below in the context of abstraction operators defined only on domains of a certain cardinality.

[7] That is, if there are actually 43 things. There are other potential problems with expressions which are universal quantification on domains with more than 43 things and existential otherwise.

Given a function f from S_i to S_j, $\pi^+(f)$ is the function composed of π^+, f, and the inverse of π^+: $\pi^+ \circ (f \circ \pi^{+^{-1}})$. It is easy to check that this is the function g from $\pi^+(S_i)$ to $\pi^+(S_j)$ such that

$$g(\pi^+(x)) = \pi^+(y) \text{ iff } f(x) = y.$$

When f is a characteristic function, then we can simplify the above since π^+ is constant on t. Writing C_f for $\{\alpha \,|\, f(\alpha) = T\}$,

$$\pi^+(C_f) = \pi^+(\{\alpha, \beta, \ldots\}) = \{\pi^+(\alpha), \pi^+(\beta), \ldots\}.$$

We will say that σ^D is *invariant under* π iff $\pi^+(\sigma^D) = \sigma^D$ and σ^D is *permutation invariant* iff σ^D is invariant under π for every permutation π on D. We can now formulate Tarski's criterion of logicality.

Tarski's Criterion of Logicality. *A notion is logical just in case on every domain it denotes a permutation-invariant member of the type-hierarchy of that domain*[8].

This is not quite right — a quantifier W that denotes the operation of existential quantification on domains containing wombats and universal quantification otherwise is sensitive to the characteristics of the particular individuals making up the domain, but passes Tarski's criterion with flying colors.[9] We can fix this problem and others like it by moving from invariance under permutations of a domain to invariance under isomorphisms between domains.[10]

As with permutations, given an isomorphism ζ from D to D' we can induce a function ζ^+ from the type-hierarchy over D to the type-hierarchy over D' exactly as above. We will say that σ^D is *invariant under* $\zeta : D \to D'$ iff $\zeta^+(\sigma^D) = \sigma^{D'}$ and σ^D is *isomorphism invariant* iff σ^D is invariant under ζ for every isomorphism ζ with domain D. A notion σ is isomorphism invariant if σ^D is isomorphism invariant for every domain D.

Tarskian Criterion of Logicality. *A notion is logical just in case on every domain it denotes an isomorphism-invariant member of the type-hierarchy of that domain.*

[8] This account, developed in the posthumous Tarski, A. (1986), was anticipated by Mautner, F.I. (1946).

[9] The example is inspired by McGee's discussion of wombat disjunction in McGee, V. (1996).

[10] An isomorphism between domains is simply a bijection or one-to-one correspondence. Because of this, the property of isomorphism invariance is sometimes called invariance under bijections. Note that one-to-one correspondences between domains typically do not extend to isomorphisms between *structures* in the model-theoretic sense.

The notions denoted by the usual logical constants are isomorphism invariant. Consider the operation of universal quantification. If ζ is an isomorphism between D and D' then

$$\{\zeta(d) \mid d \in D\} = D' \text{ so } \zeta^+(\forall^D) = \{D'\} = \forall^{D'}.$$

So the operation of universal quantification is isomorphism invariant. A little work confirms that all the usual logical constants are isomorphism invariant.[11] Consider now the relation $\prec$ which holds between $a, b \in D$ iff a is less than or equal to b and they are both positive integers. For any isomorphism ζ from $\mathbb{Z}^+$ to $\mathbb{N}$, $\zeta^+(\prec^{\mathbb{Z}^+}) \neq \prec^{\mathbb{N}}$, since $0 \not\prec^{\mathbb{N}} i$ for i in $\mathbb{N}$. A little work confirms that typical non-logical expressions are not isomorphism invariant.

2.2. *Adequacy of the Isomorphism Invariance Criterion*

Isomorphism invariance enjoys widespread acceptance as a demarcation of logical notions in mathematical contexts such as abstract model theory.[12] Insofar as philosophers make use of a principled semantic criterion of logicality, the going account is that logical constants denote isomorphism-invariant notions.[13] I have no objection to taking isomorphism invariance as the criterion of logicality for notions. However, it is inadequate as a full classification of the denotations of expressions into the logical and the non-logical. The isomorphism invariance criterion only classifies denotations of expressions that can be represented as functions from domains to members

[11] All truth-functions come out as trivially logical on the invariance criterion. We can give a more nuanced account of the logical status of the truth-functions, but doing so here would be a distraction.

[12] See Barwise, J. (1974) and Lindström, P. (1966). Vaananen, J. (2004) contains a useful overview of both Barwise and Lindström's work and subsequent developments.

[13] See, for example, Kit Fine's use of invariance criteria to distinguish good from bad abstraction principles in Fine, K. (2008). Solomon Feferman (1999) and Denis Bonnay (2008) have developed more subtle variations on the isomorphism invariance criterion with the aim to exclude quantifiers like 'There are $\aleph_{18}$ many'. Such variations do not matter for the treatment of indefinite expressions like Hilbert's ε and Russell's η — my suggestion for these cases can easily be modified for any extant variation. My later treatment of abstraction operators, and, in particular, my account of the logical status of ε-abstraction operators, would need to be modified. For example, my treatment of the logical status of Hume's principle requires that = is a logical constant, which it is not on Feferman's account. The question of how my treatment of abstraction operators fares on accounts like Bonnay or Feferman's, suitably amended, would take us too far afield from my present purpose. I hope to address it elsewhere. There is also an entirely separate tradition of proof-theoretic accounts of logicality arising from Gentzen, G. (1935). See also Dummett, Michael (1993). The relationship between this latter tradition and the account discussed here is beyond the scope of this paper.

of the appropriate type in the type-hierarchy over those domains.[14] Not all expressions of natural or logico-mathematical language are usefully thought of this way. Russell's indefinite and definite description operators η and ι, Hilbert's ε operator, and abstraction operators such as 'the number of' are all cases of expressions which do not fit nicely into the framework just given. These operators have some claim to being logical constants so we should extend our framework in a principled way to allow the isomorphism invariance criterion to give a verdict on their status.

η, ι, ε, and abstraction operators are all examples of what are sometimes called "variable-binding term operators".[15] They attach to formulas with one or more free variables to form a term that denotes an object in the domain. η, for example, attaches to a formula $A(x)$ with x free to form the term $\eta.xA(x)$ which denotes an *arbitrary* object o in the domain that satisfies $A(x)$. If there is no such object, $\eta.xA(x)$ fails to denote. ε is a total version of η; $\varepsilon.xA(x)$ denotes an arbitrary object from the domain if $A(x)$ is unsatisfiable. In our present framework, variable-binding term operators have to denote functions of type $((e \Rightarrow t) \Rightarrow e)$; that is, total functions from the power set of the domain to the domain.[16] It is a trivial fact that no total function of this type in the type-hierarchy over a non-singleton domain is permutation invariant, *a fortiori* that no total function of this type is isomorphism invariant. If we force the denotation of expressions like ε into Tarski's framework, we would face the unpleasant choice of rejecting the Tarski-Mautner criterion of logicality or accepting that no variable-binding term operator denotes a logical notion. Fortunately, we do not have to face this choice. We can adapt our framework in a natural way to allow a principled and non-trivial demarcation of the logical from the non-logical variable-binding term operators.

3. Modifying the Framework

There are two problems involved in modifying our framework. The less serious problem has to do with partial functions. It was implicitly assumed above that that a member of a type $(T_1, \ldots, T_n \Rightarrow S)$ is a total function on

[14] This is not a peculiarity of my exposition of the invariance criterion. It is a common assumption in the literature on permutation invariance.

[15] The theory of variable-binding term operators is developed in Corcoran, J., Hatcher, W. and Herring, J. (1972) and da Costa, N.C.A. (1980).

[16] Of course, this is only on the presumption that a closed term like $\eta.xA(x)$ denotes, on a domain, an entity if $A(x)$ is satisfiable. If we do not require this, we could — as I discuss below — interpret η and like operators in such a way that this does not happen. On at least one way of doing this, the resulting denotation of η is isomorphism invariant. However, this way of interpreting the denotation of η does not accurately represent its intended meaning.

the denotations of $T_1, \ldots, T_n$. However, some expressions do not denote total functions. Russell's ι operator, for example, is most naturally interpreted as the partial function[17]

$$\iota^D(X) = \begin{cases} \delta & \text{if } X = \{\delta\} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We can fix this problem by expanding the type-hierarchy to allow partial functions. We interpret $(S_i \Rightarrow S_j)$ as denoting the set of functions from subsets of S_i to S_j; analogously for more complex types. This is a friendly amendment, clearly in the spirit of Tarski's approach. It does require complicating definitions of satisfaction to accommodate non-denoting expressions, but such complications are not relevant here.[18] No change is necessary for our account of isomorphism invariance and applying it to ι gives the desired result that ι is a logical operator. We will assume this amendment in the remainder of our discussion.

3.1. *Indefinite Expressions and Generalized Notions*

The more serious problem arises with operators like η and ε. The denotation of ι on a domain is the unique partial function that takes singletons to their members. The objects denoted by closed terms such as $\iota.xA(x)$ are thus determined by the domain. Indefinite expressions like Russell's η and Hilbert's ε are different. The domain does not determinately specify a single object to serve as the denotation of a closed term like $\varepsilon.xA(x)$, since the closed terms formed with ε and η denote *arbitrary* satisfiers of the formula the operators attach to. Modeling this sort of arbitrariness is not entirely straightforward. A first approach is to assign arbitrary, but definite functions of the appropriate type to ε and η on every domain. This would fail to distinguish the notion denoted by η, say, from the notion denoted by some definite expression that denotes the same function as η on every domain, but, as noted above, many distinct expressions intuitively differing in meaning will denote the same notion. A more worrisome consequence of this approach is that the notion assigned to η and ε will not be isomorphism invariant and thus will fail the Tarski-Mautner test for logicality. However,

[17] ι can be interpreted as a total function though this interpretation is not especially natural. Scott, D. (1967) does this by positing an object outside the ordinary domain of discourse for $\iota.xA(x)$ to denote when $A(x)$ is not uniquely satisfied. This is a convenient way of modeling ι for certain purposes, but it does violence to the intended meaning of this expression.

[18] Presumably we need such complications anyways to deal with the many expressions of ordinary natural and mathematical language that cannot be assumed to denote.

the intended meanings of ε and η are not sensitive to underlying characteristics of the members of the domain. Since this property was what we are trying to model with isomorphism invariance, something has gone wrong in our account of the meanings of these expressions.

On the approach just mooted, η denotes on any domain D an arbitrary partial function from the set of non-empty subsets of D to D such that for all $X \subseteq D$, $\eta^D(X) \in X$.[19] Such a function is called a 'choice function'.[20] Note that any choice function on this set would do exactly as well as any other as the denotation of η.[21] Our first attempt at specifying a denotation for η misses this fact. What the domain determines for the denotation of η is not a particular function, but rather a range of *admissible* functions which, in some sense, could serve as the denotation of η. Our account of the denotation of operators like η and ε should respect this fact. We can do so by slightly generalizing our account of the denotations of expressions.

Let a *generalized notion* be a function which sends some domains D to a set of objects of the same type as one another in the type-hierarchy over D. A *total generalized notion* is a generalized notion defined on every domain. We can take the denotation of expressions such as η and ε to be generalized notions; the former will denote the function sending D to the set of choice functions on the non-empty subsets of D, the latter to the set of choice-functions on the full power set. We can also take the denotation of expressions that are more definite to be generalized notions. What a domain determines for the denotation of ι can be seen as a range of admissible functions, but in this case there is always only one. Using σ^D now to denote the image of the generalized notion denoted by σ on D, ι^D is always the singleton containing the function from all singletons in the power set of D to their members. Let a *definite* generalized notion be one whose denotation on every domain is a singleton. An *indefinite g*eneralized notion is one whose denotation on some domains is not a singleton.[22]

[19] Strictly speaking, a function f of type $((e \Rightarrow t) \Rightarrow e)$ is a choice function if for all g of type $(e \Rightarrow t)$, $g(f(g)) = T$ if the range of g is not {F}.

[20] The denotation of ε on a domain is a slight extension of a choice function with an arbitrary member of the domain assigned to $\emptyset$ to make ε total. I will also call this a choice function to simplify my exposition. No confusion should arise.

[21] The assignment of choice functions as the denotations of operators like η or ε presumes that such operators are extensional in the sense that their application to distinct, but co-extensional formulas results in distinct complex expressions having the same denotation. This is natural given the extensional framework we are working in. We could develop a version of my account without this presumption, but this would require complicating the framework in ways that would obscure my main point. If one objects to taking quasi-natural-language expressions like ε and η as extensional, let the meaning of ε and η be regarded as stipulated. So regarded, their meaning is clear and extensional.

[22] We assume that generalized notions are undefined on domains where their denotation would otherwise be $\emptyset$. Nothing turns on how we accommodate partial generalized notions.

Since we can view a set of functions of type T as the characteristic set of a function of type $(T \Rightarrow t)$, it is tempting to identify the denotation of an expression like ε with a function of slightly higher type. The account of expressions like ε and η is not best developed in this way; we want to distinguish cleanly between the denotation of a predicate of functions of type $((e \Rightarrow t) \Rightarrow e)$ — which is of type $(((e \Rightarrow t) \Rightarrow e) \Rightarrow t)$ — and the set of choice functions assigned to an expression like ε. This means that if we want to preserve the fact that these expressions function differently at the level of their denotations, we need to distinguish between a set of objects of type T and a function of type $(T \Rightarrow t)$. And we do want to preserve this fact since we want $\varepsilon.xF(x)$ to be a referential expression.[23] The use of generalized notions allows us a formal representation of the indefiniteness of certain denotations which allows us to preserve the thought that term-forming operators like ε really are referential expressions — i.e. their semantic type is of the form "$\ldots \Rightarrow e)$" — of a certain indefinite sort.

Using generalized notions instead of Tarskian notions allows us to mark distinctions which are otherwise obscured. The indefinite generalized notion denoted by η is isomorphism invariant.

Proof. Given D, D', let ζ be an isomorphism from D to D'. We show $\zeta^{+}(\eta^{D}) = I$. Given $f \in \eta^{D}$, $\zeta^{+}(f) = \zeta^{+} \circ (f \circ \zeta^{+^{-1}})$. This is a choice function on the power set of D' and is thus in $\eta^{D'}$. So $\zeta^{+}(\eta^{D}) \subseteq \eta^{D'}$. Given $f \in \eta^{D'}$, consider $\zeta^{+^{-1}} \circ (f \circ \zeta^{+})$. This is a choice function on the power set of D and hence is in η^{D}. Taking the image of this function under ζ^{+} and resolving yields f, so $f \in \zeta^{+}(\eta^{D})$ and thus $\eta^{D'} \subseteq \zeta^{+}(\eta^{D})$. $\square$

Consider now the generalized notion $\sigma_{\leq}$ which consists of the class of functions sending non-empty subsets of a domain to the least natural number in them and some arbitrary member of the domain otherwise. $\sigma_{\leq}$ is not isomorphism invariant.

Proof. Let f be a member of $\sigma_{\leq}^{\{1,2,3\}}$. Let $\zeta(1) = 2$, $\zeta(2) = 3$, and $\zeta(3) = 1$. ζ is an automorphism on $\{1,2,3\}$, but $\zeta^{+}(\sigma_{\leq}^{\{1,2,3\}}) \neq \sigma_{\leq}^{\{1,2,3\}}$ since $\zeta^{+}(f)(\{1, 2\}) = 2 \neq 1 = f(\{1, 2\})$. $\square$

[23] We could also move up the denotations of *all* expressions, but the resulting account differs from my account only in labeling. Note that no matter how we proceed, we have to amend Tarski's framework in some fashion if we want to maintain that $\varepsilon.xF(x)$ is a well-formed referential expression which denotes something along the lines of 'the result of applying that which is denoted by ε to that which is denoted by $F(x)$' while also maintaining that the denotation of ε is isomorphism invariant. I will use my terminology in what follows, though I acknowledge that this is somewhat a matter of taste.

In both cases, the admissible functions are not themselves isomorphism invariant. But in the case of η, the set of admissible functions on any domain is isomorphism invariant. As I just showed, this is not the case for $\sigma_{\leq}$.

This formal difference between η and $\sigma_{\leq}$ tracks an intuitive difference between the admissible functions for each. One way for an admissible function to be isomorphism variant is for the image of that function under an isomorphism to not be admissible. Such is the case with the $\sigma_{\leq}$-admissible functions. Call this sort of failure *strong isomorphism variance*. Another way is for every isomorphism to take admissible functions to admissible, though not necessarily identical, functions. Such is the case with the η-admissible functions. Call this sort of failure *weak isomorphism variance*. It is strong isomorphism variance which exposes sensitivity to features of objects in the domain. Thus our conception of the logicality of the denotations of expressions ought to disallow only those generalized notions containing strongly isomorphism-variant objects.

We can extract a plausible account of logical generalized notions from the preceding discussion: A generalized notion is logical if and only if its denotation on any domain is isomorphism invariant. That is, a generalized notion is logical if the set of admissible members of the type-hierarchy over D that it denotes on D is isomorphism invariant.[24] It is immediate that a generalized notion is logical if and only if every admissible function in its denotation on a domain is not strongly isomorphism variant. This criterion is thus in the spirit of Tarski's proposal, but allows us to classify the denotations of indefinite expressions such as variable-binding term operators operators as well as definite expressions. This criterion is non-trivial since, as demonstrated above, η denotes a logical generalized notion, whereas $\sigma_{\leq}$ does not.

4. The Logical Status of some Variable-Binding Term Operators

We can now apply our criterion of the logical status of generalized notions to some cases. This will both test the adequacy of our amendment and highlight the virtues of our shift in framework. We focus on variable-binding term operators since they are the most salient expressions left out by the earlier account of logical notions. The model theory for variable-binding term operators has been developed both under the assumption that no variable-binding term operator is logical and under the assumption that all

[24] Note that on this way of describing generalized notions, there is only one admissible member of the appropriate type in the type-hierarchy over a domain for definite predicates like 'cat' — the function which takes cats in the domain to T.

are.[25] It is somewhat surprising that no one has attempted a principled demarcation of the logical variable-binding term operators from the non-logical since the criterion just given is a natural extension of a well-known demarcation of logicality.[26] We will start with ε for reasons which will become apparent in our discussion of the abstraction operator 'the number of'.

4.1. *The Logical Status of ε*

The ε operator is governed by the laws

$$F(x) \rightarrow F(\varepsilon.xF(x)) \tag{I}$$

$$\forall x(F(x) \leftrightarrow G(x)) \rightarrow \varepsilon.xF(x) = \varepsilon.xG(x)^{27} \tag{E}$$

Terms like '$\varepsilon.xA(x)$' are to be interpreted similarly to the *referential* expression 'an object such that if anything is A, it is one'. The use of the indefinite English expression 'an' in explicating the intended meaning of ε is crucial — ε is an operator of indefinite choice.

Some indefinite variable-binding term operators like ε can be contextually eliminated by quantifying over functions. Since we can express that a function is a choice function using only higher-order logical vocabulary, we can rewrite $\phi(\varepsilon.x\phi(x))$ as

$$\exists f\ [f \text{ is a choice function and } \phi(f(\phi(x)))]$$

Some linguists have attempted to account for indefinite descriptions in natural language this way.[28] This may be the correct way to treat natural

[25] Corcoran, J. and Hatcher, W. and Herring, J. (1972) develops model-theoretic account for variable-binding term operators on which they are non-logical, da Costa, N.C.A. (1980) one on which they are all logical.

[26] The closest anyone has come to my suggestion that we shift from notions to generalized notions is Newton da Costa who, in his (da Costa, N.C.A. 1980), associates each variable-binding term operator with a "smooth operator" which is something very much like a generalized notion. However, the role of smooth operators is merely to restrict the possible denotations of expressions like ε. On particular models, each variable-binding term operator denotes some particular member of the smooth operator associated with a variable-binding term operator. His use of smooth operators to account for the meaning of this class of expressions is limited to the stray remark that smooth operators "(are) in some sense the semantical meaning of a vbto (variable-binding term operator)" (134). Since he is attempting to prove standard model-theoretic results for a theory including variable-binding term-operators, his rather definite account of the denotations of indefinite expressions is not entirely surprising since the assignment of a definite value to expressions like ε simplifies their treatment.

[27] Given an operator σ that obeys I but not E, we can define ε in higher-order logic by means of ι. Let $\varepsilon.x\phi(x) =_{df} \iota.y\exists X(\forall x(Xx \leftrightarrow \phi(x)) \wedge y = \sigma.xXx)$. Thanks to John Burgess for this point.

[28] See Reinhart, T. (1997) for a careful development of this sort of account of indefinite descriptions. I will bracket the question of whether or not the sort of account I give of the

language indefinite articles like 'a', but it seems drastic for ε since ε can be conservatively added to first-order logic whereas adding quantification over functions to first-order logic is extremely non-conservative. This situation is similar to the case of identity. We can contextually eliminate = in second-order logic, but this does not show that we are covertly engaging in higher-order quantification when we make an identity claim.[29]

I should also guard the reader against a potential misunderstanding. The arbitrariness of the intended interpretation of indefinite expressions like ε and η is not merely epistemic. We do not understand $\varepsilon.xF(x)$ as being some particular F whose identity is determined by its domain of application in some way we are blocked from knowing. Rather, it is essential to understanding an indefinite expression like $\varepsilon.xF(x)$ that we recognize that its value really is arbitrary in the sense that facts about the domain do not determine which F, if any, it denotes.[30] We should not attempt to explain away the indefiniteness of expressions like $\varepsilon.xF(x)$ by making them covertly definite. Assigning generalized notions as the denotations of indefinite expressions is a broad-brush though extensionally adequate account of their meaning. It serves to bring indefinite expressions into the fold as the sort of expressions whose denotations can be assessed for logicality.

denotation of a formal expression like ε is plausible as the basis for an account of English indefinites such as the 'A' in 'A student passed the exam.' since the complexities of the semantics of natural language indefinites is well known. Some analyses, such as Kratzer, A. (1998), *do* interpret indefinite expressions as denoting choice functions on a domain. Kratzer's analysis amounts to interpreting 'If a student passes, I'll be thrilled' as 'If f (being a student) passes, I'll be thrilled' where f is a free variable assigned a choice function relative to the background conversational context. Such a choice function can be more or less specific, of course, and this is a desirable feature since it is plausible that there are both specific and unspecific uses of indefinites in natural language (Fodor, J.D. and Sag, I.A. 1982). On Kratzer's view, the intended interpretation of ε would be roughly equivalent to the the indefinite 'a/an' in the maximally unspecific conversational context.

[29] Note also that we would also need to extend our quantificational apparatus even further to contextually eliminate ε operators of higher type.

[30] After writing the above, I discovered that Ofra Magidor and Wylie Breckenridge have recently suggested an epistemic interpretation of the arbitrariness in claims like 'Let a be an arbitrary F' (Breckenridge, W. and Magidor, O. 2012). They hold that the totality of facts about a domain (including primitive semantic facts) determines the meaning of expressions much like $\varepsilon.x\phi(x)$ on that domain. Though I am not sure I fully understand their proposal, it seems untenable: if I were to pick a marble out of a sack of indiscernible marbles, dub it 'Charlie', and replace it, wondering which marble Charlie is would not be senseless, though it would not be sensible. Likewise, if an omniscient being knows, but I cannot, which object $\varepsilon.xF(x)$ denotes, then it is silly, but not senseless, to wonder which. The same cannot be said about wondering which F I picked out with 'Let a be an arbitrary F'. Their sin is one of insufficient boldness. They should refuse to let any facts, even "primitive semantic facts", determine the value of an arbitrarily chosen object. It is worth mentioning that the isomorphism invariance criterion misclassifies indefinite expressions even if we adopt their view (which can be modeled by assigning an indefinite term like ε to a fixed but arbitrarily chosen choice function on the domain). Thus they should welcome the amendment I suggest.

A trivial modification of the above proof that the generalized notion denoted by η is isomorphism invariant shows that the generalized notion denoted by ε is also isomorphism invariant. We further note that there are two obvious extensions of standard ways of evaluating the denotations of complex expressions on a domain to a language containing ε terms on the present account of their meaning. The first pushes the arbitrariness of ε back into the metalanguage, assessing the value of sentences containing ε terms relative to an arbitrary choice function. The second eliminates arbitrariness in the metalanguage, assessing the denotations of sentences containing ε terms relative to all choice functions. Choosing between these candidates is an ideological matter which I cannot enter into here; it is enough to point out that either account can be developed so that the laws I and E are validated without disrespecting the intended reading of ε.[31] However, to fix ideas, I will briefly sketch how the latter option would work.

The principle requirement of an account of ε is that it preserve the sense in which ε is an indefinite expression. Let an ε-specification of a domain D be a pair $\langle D, \mathfrak{f}\rangle$ where $\mathfrak{f}$ is a choice function on the power set of D. For ε-specifications, we can work with notions instead of generalized notions. For any expression ρ not containing an ε term, writing ρ^D for the value of ρ on D, we let $\rho^{\langle D,\mathfrak{f}\rangle} = \rho^D$. So, for example, if $\phi = Fa$,

$$Fa^{\langle D,\mathfrak{f}\rangle} = Fa^D = F^D(a^D)^{32}$$

We evaluate ε terms on an ε-specification of D as follows

$$\varepsilon.x\phi^{\langle D,\mathfrak{f}\rangle} = \mathfrak{f}(\phi^{\langle D,\mathfrak{f}\rangle})^{33}$$

This, essentially, is to treat ε as a function constant when evaluating it on an ε-specification. Given an expression ϕ, we let

$$\phi^D = \{\phi^{\langle D,\mathfrak{f}\rangle} : \mathfrak{f} \in \varepsilon^D\}$$

[31] It is worth briefly noting that we could *still* motivate a generalized criterion of logicality even if we accepted, which I do not, a restriction to assigning notions as the denotations of expressions on domains. We would need an account of an *admissible denotation* for an expression on a domain which captures the sense in which distinct choice functions on the domain are equally good choices to assign as the denotation of ε on that domain. We would then say that an expression σ is logical (in our generalized sense) if, for all domains D, D', the set

$$\{i^+(\sigma^D) : i \text{ is an isomorphism from } D \text{ to } D'\}$$

consists of all and only the admissible denotations for σ on D'. It can easily be seen that this agrees with our above classification. It is, however, much less natural as the restriction to notions gives a misleadingly precise account of indefinite expressions.

[32] We are following tradition in playing a bit fast and loose with the difference between expressions and what they denote. This should cause no confusion.

[33] Strictly speaking, we should say that $\varepsilon.x\phi^{\langle D,\mathfrak{f}\rangle} = \mathfrak{f}(\overline{\phi^{\langle D,\mathfrak{f}\rangle}})$ where $\overline{\phi^{\langle D,\mathfrak{f}\rangle}}$ is the function of type $(e \Rightarrow t)$ in the type-hierarchy over D such that $\overline{\phi^{\langle D,\mathfrak{f}\rangle}}(d) = T$ if and only if d satisfies '$A(x)$' relative to $\langle D, \mathfrak{f}\rangle$. Such complications involving variable-binding are being ignored for comprehensibility.

The result is that the denotation of an expression on a domain is the set consisting of the denotations of that expression on all ε-specifications on that domain. When ϕ is a referential expression — an expression whose denotation would typically be a member of D — we have the possibility that ϕ^D is a non-singleton set. This indicates that the value of ϕ on D is indefinite. When ϕ is an expression whose denotation would typically be a truth-value, we have the possibility that ϕ^D is $\{\mathrm{T}\}$, $\{\mathrm{T}, \mathrm{F}\}$, or $\{\mathrm{F}\}$.[34] We say that ϕ^D is true on a domain D if $\phi^D = \{T\}$ and false otherwise. The result is that expressions involving epsilon terms are generally true only when they are true on *every specification*. So, for example, $[\lambda.x(Prime(x) \vee Composite(x))\varepsilon.y\ y = y]^{\mathbb{N}} = \{T\}$ and hence is true, yet $[\lambda.x(Prime(x)\varepsilon.y\ y = y]^{\mathbb{N}} = \{T,F\} = [\lambda.x(Composite(x)\varepsilon.y\ y = y]^{\mathbb{N}}$ and hence neither is true.

This is what we ought to expect if $\varepsilon.y\ y = y$ really is indefinite; it is definitely prime or composite, but not definitely prime or definitely composite. The resulting semantics is akin to a supervaluational semantics where we identify truth with truth on every ε-specification. We can generalize this style of account easily to accommodate any indefinite specification, letting a specification simpliciter be a domain supplemented with a choice of definite denotation for every indefinite notion. The non-classicality of this semantics is mild — like standard supervaluational semantics, it is conservative over the base classical semantics, merely allowing for the indefinite notions which were blocked on the base semantics.

4.1.1. ε is *a Logical Constant*

Still, we may worry that ε is not truly a logical constant and thus that our account of logical generalized notions overgenerates, marking as logical some indefinite generalized notions that ought to be marked non-logical. The best response to this worry is to point out the naturalness of the criterion, the fact that it correctly classifies the denotations of expressions that are plausibly logical, and the fact that it extends Tarski's thought that logical notions are insensitive to characteristics of the underlying domain to the more general case. I have argued for the first and third parts of this response already, but to shore up the second I note a number of reasons to hold that ε is a logical constant. I can see at least five such reasons:

(1) There is historical precedent for viewing the ε symbol as a logical constant. We find ε treated as a logical operator by Hilbert, Carnap, and Bourbaki.[35]

[34] We are ignoring the complication of partial notions.

[35] See Bourbaki, N. (2004), Carnap, R. (1961) and Hilbert, D. and Ackermann, W. (1939). Of course, these mathematicians and philosophers had varied attitudes to the importance of separating logical from mathematical vocabulary. Carnap explicitly argues that ε is a logical constant, albeit a non-standard one.

(2) The natural language expressions we formalize with η and ι are so closely related that to mark ι as logical (which requires merely allowing for partial Tarskian notions) without so marking η would be rather implausible. Since ε is simply η brute-forced into a total function, it would likewise be implausible to count η as logical without so counting ε.

(3) I and E can be conservatively added to the standard proof rules or axioms for standard first-order logic. That is, adding the proof rules or axioms for ε to a standard deductive system for first-order logic does not allow us to prove any ε-free sentence we could not already prove.[36] ε thus satisfies Nuel Belnap's widely accepted existence criteria for logical constants. You might think, following Restall, G. (2010), that ε should satisfy the additional requirement that given another operator τ obeying I and E, $\varepsilon.x\phi(x)$ should be identical to $\tau.x\psi(x)$ when $\phi(x)$ and $\psi(x)$ are co-extensional. ε dramatically fails this requirement. Is this problematic? No, since to impose the stronger requirement is tantamount to requiring that ε be definite. Such a demand is entirely inappropriate.

(4) It is plausible that we tacitly assume in our ordinary mathematical discourse the acceptability of indefinite expressions that function similarly to ε. Consider the practice of using expressions like 'Let a be an F' in the course of proving a generalization. We intend a to pick out an arbitrary F and treat a afterwards as a referential expression. In constructing formal proofs, we mirror this practice with the use of eigenvariables (sometimes called 'dummy names').[37] As Kit Fine notes, the epicycles we go through to eliminate eigenvariables in our formal proofs do little justice to how we actually reason and as he and Allen Hazen both note, students do better constructing

[36] Given the completeness of first-order logic, this means that we also do not extend the first-order consequence relation by the addition of I and E. Of course, ε is not conservative over every base theory. ε is not conservative over ZF when we allow ε-terms inside of the separation schema — we can then prove a version of the axiom of choice. This is to be expected; the epsilon calculus is more expressive than standard first-order logic. The culprit, however, is not ε, but rather the underlying unexploited strength of separation. See also fn. 57.

[37] Note that I am not claiming that we are forced to interpret our practice as involving a notion like ε and I am certainly not claiming that we have to interpret our formal use of eigenvariables this way. We can, for example, regard the formal use of eigenvariables as a mere technical convenience in formal inference. The informal use of expressions like 'Let a be an F' is more difficult, but some eliminative story could surely be told. My claim is rather that interpreting our ordinary use of 'Let a be an F' in terms of indefinite expressions like ε is natural. It provides a useful rational reconstruction of what we are doing when we say something like 'Let a be an F' and then go on to talk about a. The instrumental use of eigenvariables provides another such rational reconstruction, but it is less elegant and does not do justice to our actual practice.

proofs when they are taught to interpret eigenvariables as denoting objects arbitrarily chosen.[38] If we take mathematical discourse at face value, we ought to allow the use of indefinite choice. This does not yet speak directly to the logicality of ε, but to the acceptability of the arbitrary interpretation of ε. However, our use of these expressions makes no special claim about the nature of the domain or the content of the premises or conclusion of the particular proof in which they are employed. We can thus regard the practice of choosing of arbitrary satisfiers of formulas as part of the framework of proofs just as we can so regard the quantifiers and the connectives.[39] Being part of the framework of proofs in this sense is a plausible proof-theoretic criterion of the logicality of expressions.[40]

(5) Though it is not at all plausible that all the components of a definition of a logical constant are themselves logical, nevertheless, as pointed out by Neil Tennant, it is plausible that if we can define σ by means of a definition employing σ' as the *sole* primitive expression, then if σ is logical, so too is σ'. Applying this to the case of ε, we note that we can implicitly define $\exists$ by means of ε (in the presence of I and E) with the schema $\exists x\phi(x) \leftrightarrow \phi(\varepsilon.x\phi(x))$.[41]

This concludes my case for the logical status of the expression ε. Each of the above is independently compelling; jointly they constitute a strong case for including ε in our logical vocabulary. Even stronger cases can be mounted for ι and η. If the expressions ε, ι, and η are logical, then we should expect their denotations to satisfy our criterion of logicality. The criterion of logicality for generalized notions gets this exactly right. In the absence of plausible examples of non-logical expressions whose denotations are marked logical by the criterion I have given, we need not be worried that the above criterion overgenerates. Of course, abstraction operators like 'the number of' are also variable-binding term operators, and we might start to worry if our criterion marked the denotation of abstraction operators as logical. This case is more complicated and deserves fuller treatment.

[38] Fine introduces arbitrary objects to correspond to our talk about objects arbitrarily chosen (Fine, K. 1985). Increasing one's ontology this way seems less preferable to increasing one's ideology with indefinite expressions so as to mirror ordinary reasoning.

[39] Hazen, A. (1987) shows how to use ε to replace the use of eigenvariables in a proof-theoretic setting.

[40] See Dosen, K. (1989) for an account of this criterion for the case of formal deductions. A similar point can be made with respect to the natural language expressions formalized with η, ι, and ε. As stressed to me by Paul Egré, these expressions have the surface appearance of functional expressions like quantifiers rather than substantive expressions.

[41] See Tennant, N. (1980) for a version of this argument set in a natural deduction context. The definition of $\exists$ in terms of ε is due to Hilbert. See Hilbert, D. and Ackermann, W. (1939).

4.2. *The Logical Status of Abstraction Operators*

Traditionally, an *abstraction operator* O is a function whose range is a subset of a domain D, whose domain is a type in the type-hierarchy over D, and which is defined by the following schema:

$$\forall \alpha, \beta\ (O(\alpha) = O(\beta) \leftrightarrow E(F, G))^{42}$$

where α, β are n^{th}-order variables and E is an equivalence relation on class over which α, β range. α and β can be first-order, as in Frege's direction principle:

The direction of x = the direction of $y \leftrightarrow x$ is parallel to y

or second-order, as in Hume's principle:

The number of F = the number of $G \leftrightarrow F$ is equinumerous with G[43] (HP)

The most interesting abstraction operators are given by higher-order *abstraction principles* like HP where E is an equivalence relation on the power set of the domain. We will focus our discussion on the second-order case. We need to modify the traditional definition for our current purpose since we can no longer presume that expressions denote notions. We will take an abstraction operator to be a generalized notion — that is, a function from domains to a class of functions, each of which satisfies the relevant abstraction principle on that domain — with an associated equivalence relation E.[44] We can see such abstraction operators as collections of ways of indexing the cells of the partition induced by E on the power set of the domain with objects from the domain. We assume that an abstraction operator is non-empty on any domain that permits the existence of a function satisfying the abstraction principle.

Some abstraction operators such as 'the number of' have been thought to be logical in *some* sense or other. This is a natural thought given the view of Crispin Wright and other neo-logicists of the Scottish variety that Hume's principle is an implicit definition of the concept of cardinal number. On this view, acceptance of an object-language sentence expressing HP suffices to confer a meaning on the expression 'the number of'.[45] It is plausible that if 'the number of' is implicitly defined by HP, then it inherits the logical

[42] The initial quantifiers will henceforth be dropped for readability where appropriate. n can be any natural number though we will restrict our attention to second-order abstraction.

[43] Both examples originate in Frege, G. (1980).

[44] Strictly speaking, E is not an equivalence relation, but a function from domains to equivalence relations on them. We simplify for purposes of comprehensibility.

[45] I will not be overly careful in distinguishing the metalanguage schema HP from the object language sentence expressing it.

status of HP's right-hand side.[46] Others, such as Aldo Antonelli, have criticized this sort of claim by arguing that the meaning conferred on abstraction operators like 'the number of' is not permutation invariant, even though the relation of equinumerosity is.[47] Antonelli's criticism is cogent within the Tarskian framework he is working in, but it is not immediately obvious what we should say about this objection within the amended framework we have adopted.

We can give a precise account of exactly which abstraction operators are isomorphism invariant and thus logical though we have set aside one way of understanding the role of abstraction principles. Any view on which the acceptance of abstraction principles somehow introduces or brings into existence the objects that are the range of the functions comprising an abstraction operator must be treated, if at all, in a way that does not pay proper attention to the introduction of these objects.[48] The trouble is that our criterion makes no provision for the genesis of the objects making up a domain, treating all objects comprising it on a par. In light of this, our discussion is restricted to views on which the functions in an abstraction operator take as their range some subset of the domain under consideration.[49]

Let an abstraction operator σ be *full* if, for every domain D, σ^D contains every admissible function. That is, a function f is in σ^D if and only if non-E-equivalent members of the power set of D are assigned non-identical objects in D by f and equivalent members identical objects.

Lemma. *An abstraction operator is isomorphism invariant only if it is full.*

Proof. Let σ be a non-full abstraction operator with associated equivalence relation E. There is then, on some domain D, a function $f \notin \sigma^D$ from the power set of D into D which respects E. Let g be a member of σ^D. $g \in \sigma^D$, so g respects E. Since $|D \setminus Ran(g)| = |D \setminus Ran(f)|$, there is a bijection ζ from $D \setminus Ran(g)$ to $D \setminus Ran(f)$. Since g and f respect E, $g(A) \neq g(B)$ if and only if A and B are not E-equivalent if and only if $f(A) \neq f(B)$. So, for each

[46] Of course it is rather difficult to maintain that HP implicitly defines a *notion* since there is no unique function that satisfies it. This has led Wright and others to weaken the standard uniqueness criterion for successful implicit definitions. As will be seen below, we can make better sense of HP as an implicit definition once we switch to our amended framework.

[47] See Antonelli, G.A. (2010). I note that Antonelli's paper inspired me to generalize the Tarskian criterion of logicality and that I found his objections important and provocative.

[48] We find the strenuous rejection of this picture of abstraction principles in Antonelli, G.A. (2010). I am in full agreement with Antonelli that this picture is mysterious at best. We further agree that the clearest way of viewing abstraction operators is as indexings of the partition on the power set of a domain D with indices drawn from D.

[49] We can treat such views partially by assuming that the domain with which we assess the logical status of operators like 'the number of' is that which results from the acceptance of abstraction principles. The logical status of the *action* of expanding the domain in this way cannot be treated here.

E-equivalence class $[A]$, there is a unique member of $Ran(f)$ and a unique member of $Ran(g)$ mapped to its members by f and g respectively. Given this, we extend ζ to an automorphism by setting $\zeta'(g(A))$ to $f(A)$ for every A in the power set of D. $\zeta'^{+}(g) = f$, but $f \notin \sigma^D$, so σ is not invariant under ζ' and thus not isomorphism invariant. □

It is an almost immediate corollary of this lemma that logical abstraction operators are indefinite.[50] This is not surprising; non-full abstraction operators differentiate between members of a domain. They thus violate the intuitive constraint on logical notions which underwrites the isomorphism invariance criterion of logicality. In contrast, it is to be expected that full abstraction operators are often isomorphism invariant and hence logical. This is the case for operators like 'the number of' whose associated equivalence relation (henceforth abbreviated $\approx$) is isomorphism invariant.[51] We thus have an almost converse to the above lemma.

Lemma. *A full abstraction operator is isomorphism invariant if its associated equivalence relation is isomorphism invariant.*

Proof. Consider a full abstraction operator σ whose equivalence relation E is isomorphism invariant. Let D, D' be isomorphic domains on which σ^D and $\sigma^{D'}$ are non-empty and ζ an isomorphism from D to D'. Remember that ζ^+ is the extension of ζ to the entire type-hierarchy over D. Since E is isomorphism invariant, $\zeta^+(E^D) = E^{D'}$. Suppose $f \in \sigma^D$.

$$\begin{aligned}
\zeta^+(f)(\zeta^+(S)) = \zeta^+(f)(\zeta^+(T)) &\Leftrightarrow \zeta^+(f(S)) = \zeta^+(f(T)) && (\textit{df.}\ \text{of}\ \zeta^+)\\
&\Leftrightarrow f(S) = f(T) && (\zeta^+\ \text{preserves}\ =)\\
&\Leftrightarrow E^D(S,T) && (f \in \sigma^D)\\
&\Leftrightarrow E^{D'}(\zeta^+(S), \zeta^+(T)) && (\zeta^+(E^D) = E^{D'})
\end{aligned}$$

So $\zeta^+(f) \in \sigma^{D'}$. Conversely, given $g \in \sigma^{D'}$, there is an $f \in \sigma^D$ such that $\zeta^+(f) = g$. So $\zeta^+(\sigma^D) = \sigma^{D'}$. □

Any abstraction principle whose equivalence relation is isomorphism invariant defines an isomorphism-invariant, hence logical, abstraction operator *when* we take it to denote the corresponding full generalized notion. In fact, the only logical abstraction operators are those with isomorphism-invariant associated equivalence relations.

[50] We only need the fact that any respectable equivalence relation can be respected by more than one indexing with members of the underlying domain.

[51] $\approx$ is not only isomorphism invariant, but expressible in purely logical vocabulary. For example, we can express $F \approx G$ thus:

$$\exists f[\forall x(F(x) \to G(f(x))) \wedge \forall y(G(y) \to \exists! x[F(x) \wedge f(x) = y])]$$

Lemma. *An abstraction operator is isomorphism invariant only if its associated equivalence relation is isomorphism invariant.*

Proof. Let σ be an isomorphism-invariant abstraction operator and D, D' isomorphic domains on which σ is non-empty. Given an isomorphism ζ from D, D', $\zeta^+(\sigma^D) = \sigma^{D'}$. Let $f \in \sigma^D$. We show that $\zeta^+(E^D) = E^{D'}$ as follows.

$$\begin{aligned} E^D(S,T) &\Leftrightarrow f(S) = f(T) && (f \in \sigma^D) \\ &\Leftrightarrow \zeta^+(f(S)) = \zeta^+(f(T)) && (\zeta^+ \text{ preserves } =) \\ &\Leftrightarrow \zeta^+(f)(\zeta^+(S)) = \zeta^+(f)(\zeta^+(T)) && (\textit{df.} \text{ of } \zeta^+) \\ &\Leftrightarrow E^{D'}(\zeta^+(S), \zeta^+(T)) && (\zeta^+(\sigma^D) = \sigma^{D'}) \end{aligned}$$

Combining these three lemmas gives us a precise delineation of the logical abstraction operators.

Proposition. *An abstraction operator σ is logical if and only if it is full and its associated equivalence relation E is isomorphism invariant.*

The only isomorphism-invariant abstraction operator satisfying HP is thus the full indefinite generalized notion. In fact, all isomorphism-invariant abstraction operators are indefinite generalized notions. Such generalized notions can be seen as *arbitrary* indexings of the partition given by E exactly as we see the denotation of ε as an *arbitrary* choice function.[52] When we take abstraction operators more definitely, excluding certain otherwise admissible indexings, we are importing non-logical content and, as a result, these operators turn out to be non-logical. Our initial worry about abstraction operators like 'the number of' coming out logical on our revised criterion is thus misplaced. It is only a very special class of such operators that come out as logical — the indefinite operators — and these tell us very little about the nature of the members of the domain, treating all members of the domain alike as potential indexing devices.[53]

[52] The connection with ε can be drawn out more directly. I will show how this can be done in the next section.

[53] A helpful reviewer asks whether the logical status of abstraction operators is unchanged when we add a cross-abstraction identity principle like those considered in Fine, K. (2008) and Cook, Roy T. and Ebert, Philip A. (2005). That is, suppose we have two abstraction operators O_1 and O_2 given by abstraction principles formulated with E_1 and E_2 and the principle

$$O_1(X) = O_2(Y) \leftrightarrow \forall Z\,[E_1(X,Z) \leftrightarrow E_2(Y,Z)]$$

Given the logical status of O_1 and O_2 without such cross-abstraction identity principles, what can we say about their logical status with the additional constraint? And how should we treat the denotations of such principles given our interpretation of certain abstraction operators as indefinite? The issue is too complex to be discussed in detail here, but the upshot is that, on the most straightforward treatment, if O_1 and O_2 are both logical without cross-abstraction identity, then they are still logical with cross-abstraction identity — as far as the criterion

This result allows us to arbitrate the dispute between Wright and Antonelli alluded to above. When we have an equivalence relation like $\approx$ which is isomorphism invariant, then the domain determines an isomorphism-invariant class of functions that satisfy the corresponding abstraction principle. If abstraction operators like 'the number of' denote *notions* instead of *generalized notions*, then there is no way to assign a denotation that correlates exactly with the content given by the abstraction principle. On the other hand, on my amended account both Wright and Antonelli are right. Antonelli is right that isomorphism-variant abstraction operators are not intuitively logical as they differentiate between members of the domain. Wright is right that HP succeeds as an implicit definition of a logical expression for it determines, at least on infinite domains, a non-empty full generalized notion and one which is moreover both unique and isomorphism invariant.

Hume's principle can only succeed in defining a more definite generalized notion in the presence of background constraints on admissible functions. Such constraints undermine the logical status of 'the number of'. Without such background constraints, we can take HP either as a failed attempt to implicitly define a definite generalized notion or as a successful attempt to implicitly define an indefinite generalized notion on infinite domains. The latter option is an interesting way of understanding Hume's principle that has not been explored in the literature. We will explore this view in the next section once we have shown how to explicitly define abstraction operators using a higher-order version of ε.

Now, although we have given an account of the logical abstraction operators according to our criterion, we might still wonder if such operators are truly logical. Some logical abstraction operators will be empty on some domains since there will not be enough members of the domain to index every cell of the partition induced by E. Such is famously the case with Frege's basic law V.

(V) The extension of F = the extension of $G \leftrightarrow \forall x\,(F(x) \leftrightarrow G(x))$

The operator 'the extension of' as defined by V is empty on *every* domain. Less disastrously, the operator 'the number of' is empty on all finite domains since we need $n + 1$ distinct indices to index the equinumerosity partition of a domain of size n. Since logical constants are supposed to have universal applicability, we might want to restrict the class of logical generalized notions to those that are total — that is, to those that are non-empty on

under consideration in this paper is concerned. If exactly one is not logical without cross-abstraction identity, then enforcing the cross-abstraction identity condition can force the other to be non-logical as well. Note that enforcing this sort of condition is in tension with the intuitive picture of the meaning of abstraction operators given above. I hope to return to this very interesting issue elsewhere.

every domain. Consequently, we might want to say that though the full generalized notion that satisfies HP is isomorphism invariant, it is nonetheless not truly logical since it is not total. This is especially pressing when we view HP as an implicit definition since on finite domains it fails, in a sense, the existence requirement on good implicit definition.[54] This additional constraint goes beyond the criterion of logicality I am addressing here and I do not want to take a definite stand on this issue — though I do want to note two things.

First, the general operation of abstraction on logical equivalence relations is truly logical. That is, the binary abstraction operator *§* given by

$$\S(E,F) = \S(E,G) \leftrightarrow E(F,G)$$

where E is restricted to isomorphism-invariant equivalence relations denotes an isomorphism-invariant total generalized notion. Using $\wp(D)$ for the power set of a domain D, $\S^D$ is a class of partial functions from $\wp(\wp(D) \times \wp(D)) \times \wp(D)$ into D. On finite domains, no member of $\S^D$ will be defined for pairs E, F where E is equinumerosity. On infinite domains, no member of $\S^D$ will be defined for pairs E, F where E is the equivalence relation of having finite symmetric difference.[55] On no domain will a member of $\S^D$ be defined for E, the equivalence relation of co-extensionality. We can view all such unary abstraction operators as cases of this binary abstraction operator where we fix the equivalence relation E. So even if some abstraction operators are not truly logical since they are not total, they can be obtained in particular domains from abstraction operators that are truly logical.

Second, for an abstraction operator σ whose equivalence relation is not only isomorphism invariant, but also expressible in logical vocabulary, we can define a total generalized notion which agrees with σ on domains where it is non-empty. We can then formulate versions of abstraction principles much like HP in entirely logical vocabulary. This construction avoids the problem with the existence requirement on implicit definitions since it is immediate that the generalized notion defined is non-empty. Since this construction is of some independent interest, we will spend a bit more time developing it.

4.3. *The Logical Status of ε-abstraction Operators*

The above arguments for the logicality of ε can be extended in a natural way to justify the logical status of ε's higher-order cousin ε' which attaches

[54] That is, there exists a class of functions satisfying HP on finite domains, but only in the trivial sense that the empty set contains all such functions.

[55] See Boolos, G. (2007). This equivalence relation is isomorphism invariant and can be expressed in entirely logical vocabulary.

to formulas with free function variables of type $((e \Rightarrow t) \Rightarrow e)$. We read an expression like $\varepsilon'.fA(f)$ as denoting an arbitrary function of that type which satisfies $A(f)$ if anything does. Given D, ε'^D is:

$$\{f \mid f : \wp(D^{\wp(D)}) \rightarrow D^{\wp(D)} \text{ where } f(S) \in S \text{ if } S \neq \varnothing\}.$$

It is easily checked that ε'^D is isomorphism invariant. Letting ε' be governed by the laws

$$A(f) \rightarrow A(\varepsilon'.fA(f)) \tag{I$'$}$$

$$\forall f(A(f) \leftrightarrow B(f)) \rightarrow \varepsilon'.fA(f) = \varepsilon'.fB(f) \tag{E$'$}$$

it can be seen that ε' conservatively extends the full standard third-order consequence relation.[56] We thus have good reason to think that ε' is a logical constant if ε is.[57] And ε is.

We can use ε' to define analogues of certain abstraction operators such as 'the number of'. Let $H(f)$ be the formula:

$$\forall F \forall G\, f(F) = f(G) \leftrightarrow F \approx G$$

where f is a function variable of type $((e \Rightarrow t) \Rightarrow e)$. $H(f)$ holds of a function g from the power set of D to D only if g indexes the equinumerosity partition of the power set of D with members of D. The following is an almost immediate consequence of I$'$:

$$H(\varepsilon'.fH(f)) \leftrightarrow \exists f H(f) \tag{Q}$$

Since I$'$ entails Q, Q is a logical truth. Expanding and rewriting '$\varepsilon'.f.H(f)$' as $\mathfrak{N}$, we obtain:

$$\forall F \forall G[\mathfrak{N}(F) = \mathfrak{N}(G) \leftrightarrow F \approx G] \leftrightarrow \exists f \forall F \forall G[f(F) = f(G) \leftrightarrow F \approx G]$$

the left-hand side of which is the familiar-looking principle

$$\mathfrak{N}(F) = \mathfrak{N}(G) \leftrightarrow F \approx G. \tag{HP^A}$$

[56] This follows from the fact that we can extend any full standard model M of a third-order language L to a model M^* of $L + \varepsilon'$ that assigns a fixed choice function of type

$$((((e \Rightarrow t) \Rightarrow e) \Rightarrow t) \Rightarrow ((e \Rightarrow t) \Rightarrow e))$$

to the symbol ε'. M^* then validates both E$'$ and I$'$.

[57] Since standard third-order logic is not complete, the applicability of these results to higher-order deductive systems is non-trivial. We can prove, however, that E$'$ and I$'$ can be conservatively added to standard deductive systems for third-order logic if we restrict the comprehension schema to formulas not containing ε'. This follows from the completeness of this deductive system for Henkin models of third-order logic. If we add all the instances of the comprehension schema, E$'$ and I$'$ suffice to derive the axiom of choice. However, we *can* conservatively add E$'$ and I$'$ to the deductive system comprised of all the comprehension schemas, the quantifier rules, and the axiom of choice.

$\mathfrak{N}^D$ is an *arbitrary function* from the power set of D into D that indexes the equivalence classes of the power set of D under $\approx$ in any domain which permits such an indexing, an arbitrary function otherwise. It is never undefined. HP^A is essentially an indefinite version of Hume's principle. Since Q is a logical truth, HP^A is a logical consequence of $\exists f H(f)$. But $\exists f H(f)$ is true in a domain if and only if the domain is infinite. Thus HP^A is a logical consequence of a statement expressing that the domain is infinite. It is sufficient to derive second-order arithmetic in higher-order logic.

On domains in which HP implicitly defines a non-empty full abstraction operator, HP^A will hold if and only if HP does. Making the further assumption that HP is false if it defines an empty abstraction operator, HP^A is logically equivalent to HP. So, in a sense, HP^A is an explicit rendition of the intended interpretation of HP. The construction is perfectly general. For any abstraction operator σ whose associated equivalence relation can be expressed in a (higher-order) language L, we can define a total generalized notion (an *ε-abstraction operator*) σ' in $L + \varepsilon'$ which agrees with σ on cases where it is non-empty. Moreover, if the equivalence relation E is expressible in purely logical vocabulary, our defined generalized notion will be truly logical, being both isomorphism invariant and defined on every domain. Since expressions definable in terms of other logical constants are intuitively logical, this is a welcome result. Of course, the left-hand portion of the instance of I$'$ defining σ' will be false in domains where σ is empty since $\mathfrak{N}^D$ will then be an arbitrary function which does not respect $\approx^D$. But this is as it should be.

For example, let $\approx'$ be the relation that holds between the Fs and the Gs if and only if the symmetric difference of the Fs and the Gs is finite. Consider the "nuisance principle" (Boolos, G. 2007)

$$\mathfrak{S}(F) = \mathfrak{S}(G) \leftrightarrow F \approx' G. \tag{NP}$$

$\mathfrak{S}$ is non-empty only in finite domains. We can give the same ε' treatment of the nuisance principle that we gave HP. Call the resulting indefinite version of the nuisance principle NP^A. NP^A, like HP^A, is the left-hand side of a biconditional logical truth Q$'$, the right-hand side of which is its existential generalization. Both HP^A and NP^A are consistent, but jointly inconsistent. Is this problematic? No. Their inconsistency rests on the fact that a domain cannot be both finite and infinite and hence there cannot be an indexing of the equivalence classes under both $\approx$ and $\approx'$. So even though Q and Q$'$ are both logical truths and expressible in purely logical vocabulary, their left-hand sides are never true together. Though, if true, HP^A remains true under every reinterpretation of its non-logical content, it is not a logical truth since it is false in finite domains. It is thus not a logical truth in the modern sense deriving from Tarski. Likewise with NP^A. The situation is similar to that obtaining between $\exists x \exists y\, x \neq y$ and $\exists x \forall y\, x = y$. The former is true only in domains containing at least two things. The latter is true in only singleton domains. Neither contains any non-logical vocabulary, but neither is a logical truth or a logical falsehood.

The ε' construction has the virtue of highlighting and improving the sug-gested indefinite interpretation of abstraction operators. ε-abstraction operators are *explicitly* arbitrary indexings of the partition induced by the equivalence relation E. ε-abstraction operators have two advantages over simple abstraction operators. First, an operator like $\mathfrak{N}^D$ is a total generalized notion, being defined on every domain. Second, HP^A is not an implicit definition as HP is, but is rather the left-hand side of a biconditional consequence of the axioms for ε', the right-hand side of which states the truth conditions for the left-hand side. We can thus avoid worrying about how the stipulated truth of HP manages to implicitly define the 'the number of' given that it does not wear its indefinite character on its sleeve and is non-empty only on infinite domains.

One payoff of our ε-abstraction construction is a defensible and novel form of structuralism about the mathematical objects.[58] Call the objects characterized using ε' *abstracts*. Abstracts are simply arbitrary indexing devices drawn from the underlying domain. They have no special properties — being just arbitrary ordinary objects — and none of their ordinary properties intrude on their role as indexing devices.[59] This form of structuralism allows us to avoid many of the objections to introducing mathematical objects with abstraction principles. Since we are not laying down implicit definitions, but rather explicitly defining ε-abstraction operators, we do not have to worry about jointly consistent, but pairwise inconsistent abstraction principles. There are such collections of abstraction principles, but the right-hand sides of the principles like Q that they are the left-hand sides of are never jointly true, so we have good reason to not accept the entire collection.[60] Likewise, if the range of distinct abstraction operators were disjoint, then accepting even jointly consistent abstraction principles could require that there be more abstracts than objects. This is problematic if we

[58] This view resembles eliminative or *in re* structuralist views rather than mystical or *ante rem* views since it does not interpret $\mathfrak{N}$ as ranging over a distinguished class of abstract objects. A full discussion of this approach to Hume's principle and how it squares with various members of the structuralist family is beyond the scope of this paper. I hope to return to this issue elsewhere.

[59] Defending this picture as an account of our concept of number cannot be attempted here. Such a defense would have to investigate whether the indefinite indexings defined by $\mathfrak{N}$ captured enough of how *we* conceive of numbers and how much is enough in this regard. This project is vastly beyond the scope of this paper.

[60] If we developed our structuralist account in terms of implicit definitions of indefinite operators as suggested in the previous section, then we would still have to deal with the bad company objection. This is a strong reason to favor ε-abstraction as the basis for this type of structuralism.

expect that abstracts are members of the domain. This is the problem of "hyperinflation" raised by Kit Fine.[61] On the structuralist view I have sketched, there is no such problem since the range of abstraction operators defined with ε' can and often do overlap.[62]

ε-abstraction principles have no special epistemological status. We are in no better position to know HP^A than we are to know its existential generalization. *Mutatis mutandis* with NP^A. This allows us to take a nuanced position on the logical character of Hume's principle. Principles like HP^A allow us to shift from talking about a partitioning of the power set of the domain to talking about representatives for each cell of the partition. Q guarantees that we can do this if there exists a mapping of cells to representatives. The existence of such a mapping is equivalent to the claim that there are suitably many or suitably few things. Since the size of the domain is a substantial fact on which logic takes no stand, the truth of HP^A is likewise a substantial fact on which logic takes no stand. We can thus disentangle the logical content of the HP^A from the substantial content. $\mathfrak{N}$ is a logical constant in the fullest sense. And

$$H(\varepsilon'.fH(f)) \leftrightarrow \exists f H(f)$$

is a logical truth. It is thus a logical fact that the claim that there are infinitely many things suffices for the truth of HP^A and consequently for second-order arithmetic interpreted in the structuralist fashion mooted above. The claim that the universe is infinite directly entails the ε-abstraction version of Hume's principle. But neither HPA nor the claims of second-order arithmetic suitably interpreted are logical truths.

Why we should care that Hume's principle and the like are purely logical if they are not logical truths? $\exists x \exists y\, x \neq y$ is, even if true and purely logical, not especially interesting. On the other hand, the ε-abstraction version of HP guarantees, on the basis of logic alone, that we can introduce referential devices corresponding to the cells of the equinumerosity partition on the power set of a domain. That is, it allows us to introduce things playing the role of — or being! — numbers on the mere basis of logic, given the

[61] See Fine, K. (2008), p. 6. It is worth noting that Fine very briefly discusses the costs and benefits of adopting a variable 'the number of' operator (*op. cit.* pg. 25, fn. 13) though it is not entirely clear what sort of operator he has in mind.

[62] This may remind the reader of the so-called "Caesar" problem. On the ε-abstraction account of HP^A, this problem is misguided. If there is a function that satisfies $H(f)$, then there is a function g satisfying $H(f)$ such that $g(F)$ is the object denoted by b. So we can use the famed conqueror of Gaul as a representative if there are enough additional things. On the other hand, no statement of the form 'The number of F is b' where b is a constant term denoting an object in our domain will be provable or refutable on the basis of the definition of $\mathfrak{N}$, the axioms governing ε', and our background logic. And this is how it should be given the view that numbers are arbitrary indexing devices.

non-logically true, but purely logical fact that the universe is infinite. The fact that this version of Hume's principle is purely logical is actually slightly misleading way of describing what is so interesting about it. The most important fact is that the equivalence of this principle with the claim that the universe is infinite is a logical truth in the modern — and, obviously, the older — sense.

Of course, they would be logical truths in the older account mentioned in my discussion of Tarski's account of logical truth and logical consequence. As I mentioned above, I prefer and work with the modern account, but I do not argue for it here. For this older account, my result is even better: Hume's principle, suitably formulated with ε-abstraction principles, is an obvious logical truth if the universe happens to be infinite. And suitably formulated principles are logical truths if the universe happens to be finite. If we adopt the structuralist viewpoint suggested above, we can do a substantial amount of mathematics in pure logic without abandoning the thought that numerical expressions like 'the number of trees in my yard' are *referential* expressions without taking on additional commitments to problematic ontology. What we need is merely that indefinite expressions like ε and ε' are logical expressions and complex terms like $\mathfrak{N}(F)$ are logical — yet still referential — expressions.

5. Conclusion

With the exception of partial generalized notions like ι, all generalized notions newly classified as logical are indefinite in the sense defined above. Accepting indefinite generalized notions as our account of the denotations of expressions like ε or η amounts to an expansion of our ideology. It is not entirely dissimilar to the now widely accepted increase in ideology obtained by accepting irreducibly plural quantification.[63] Accepting this new ideology allows us to give a semantic account of certain indefinite expressions without perverting their intended meaning. The new ideology is useful as well, providing new ways of interpreting certain infamous abstraction operators as well as allowing the direct construction of ε-abstraction operators.

The naturalness of both the suggested amendment of the framework and the resulting criterion of logicality, the ability to adequately represent the meaning of indefinite expressions like ε and η, and the additional understanding of abstraction operators and their ε-abstraction correlates are sufficient reason to amend Tarski's framework. The switch to generalized notions vastly improves our understanding of the logical character of the indefinite expressions that figure in our informal logical practice. As with Tarski's

[63] Irreducibly plural quantification was initially advocated in Boolos, G. (1984).

original account, our broadened invariance criterion for generalized notions yields a plausible necessary condition for being a logical constant–logical constants denote isomorphism-invariant generalized notions. This, in turn, yields an improved account of logical truth and consequence, allowing us to better represent the logical relations among various claims involving indefinite expressions.

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